Non-symplectic automorphisms of $K3$ surfaces

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Outline

Introduction

Definition

X: compact complex surface.

$$X\text{: } K3 \overset{\text{def}}{\iff} \begin{cases} K_X \sim 0 \\ \dim H^1(X, \mathcal{O}_X) = 0 \end{cases}$$

Example

- $X : X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0 \subset \mathbb{P}^3$
- $X \xrightarrow{2:1} \mathbb{P}^2 \supset$ (non-singular sextic curve)

Proposition

$\dim H^{2,0}(X) = \dim H^{0,2}(X) = 1, \dim H^{1,1}(X) = 20.$

X has a nowhere vanishing holomorphic 2-form.

$$H^{2,0}(X) = \mathbb{C}\langle \omega_X \rangle$$

Proposition

$H^2(X, \mathbb{Z})$ has a structure of a lattice by the cup product.

Definition

- $S_X := \{x \in H^2(X, \mathbb{Z}) | \langle x, \omega_X \rangle = 0\}$
  Néron-Severi lattice
- $T_X := S_X^{\perp}$ in $H^2(X, \mathbb{Z})$
  transcendental lattice

$$1 \leq \operatorname{rank} S_X \leq 20, \quad 2 \leq \operatorname{rank} T_X \leq 21.$$

Automorphisms of $K3$ surfaces

G: finite subgroup of Aut (X).
$g \in G$, $g^*\omega_X = \alpha(g)\omega_X$ where $\alpha(g) \in \mathbb{C}^\times$.
We have a homomorphism $\alpha : G \to \mathbb{C}^\times$ and an exact sequence $1 \to \operatorname{Ker}\alpha \to G \xrightarrow{\alpha} \mathbb{Z}/I\mathbb{Z} \to 1$.

Example

$X : X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0 \subset \mathbb{P}^3$.

- The permutations of coordinates.
- $X_i \mapsto \sqrt{-1}X_i$.

$$G := \mathfrak{S}_4 \ltimes (\mathbb{Z}/4\mathbb{Z})^3 \text{ acts on } X.$$
$$1 \to \mathfrak{S}_4 \ltimes (\mathbb{Z}/4\mathbb{Z})^2 \to G \xrightarrow{\alpha} \mathbb{Z}/4\mathbb{Z} \to 1.$$

Example

$X \xrightarrow{2:1} \mathbb{P}^2 \supset$ (non-singular sextic curve)

- The covering transformation induces an automorphism ι on X.
- $\iota^* \omega_X = -\omega_X$.

$$1 \to 1 \to G := \langle \iota \rangle \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z} \to 1.$$

Definition

$$1 \to \operatorname{Ker} \alpha \to G \xrightarrow{\alpha} \mathbb{Z}/I\mathbb{Z} \to 1$$

$$G : \text{symplectic} \overset{\text{def}}{\iff} I = 1$$

Theorem (Mukai)

For a finite group G, the following two conditions are equivalent to each other:

- G has a symplectic action on X.
- G has an embedding $G \subset M_{23}$ into the Mathieu group and decomposes $\{1, 2, 3, \dots, 24\}$ into at least 5 orbits.

Today, we study non-symplectic cases ($I \neq 1$).

Proposition (Nikulin, Xiao, $\cdots$)

- Φ :Euler function $\Longrightarrow \Phi(I) | \operatorname{rank} T_X \leq 21$.
- $I \neq 60$. $(\Phi(60) = 16)$

Non-symplectic automorphisms

Definition

σ: automorphism of order I on X.

$$\sigma : \text{non-symplectic} \overset{\text{def}}{\iff} \sigma^* \omega_X = \zeta_I \omega_X$$

where ζ_I is a primitive I-th root of unity.

Proposition (Nikulin)

Let ι be a non-symplectic involution on X.

$$X^\iota := \text{Fix}(\iota) = \begin{cases} \phi \\ C^{(1)} \amalg C^{(1)} \\ C^{(g)} \amalg \mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1 \end{cases}$$

where $C^{(g)}$ is a non-singular curve with genus g.

Proposition (Vorontsov, Kondo)

Assume that σ acts on trivially on S_X.

- If $\mathbf{ord}\,\sigma$ is prime-power then $\mathbf{ord}\,\sigma = p^k = 2^\alpha, 3^\beta, 5, 5^2, 7, 11, 13, 17, 19$ $(1 \leq \alpha \leq 4, 1 \leq \beta \leq 3)$ and S_X is p-elementary.

- If $\mathbf{ord}\,\sigma$ is non-prime-power then S_X is unimodular.
  - If $\mathbf{rank}\,S_X = 2$ then $\mathbf{ord}\,\sigma | 66, 44$ or 12
  - If $\mathbf{rank}\,S_X = 10$ then $\mathbf{ord}\,\sigma | 42, 36$ or 28
  - If $\mathbf{rank}\,S_X = 18$ then $\mathbf{ord}\,\sigma | 12$

  The cases were studied by Kondo.

p-elementary lattices

Definition

L: lattice, p: prime number.

$L^* := \mathrm{Hom}(L, \mathbb{Z})$.

L: p-elementary $\overset{\text{def}}{\iff} L^*/L = (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$.

Example

- $A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ is a 3-elementary lattice with $a = 1$.
- D_4 is a 2-elementary lattice with $a = 2$.
- $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a p-elementary lattice with $a = 0$.

Theorem (Nikulin)

ι: non-symplectic involution acting trivially on S_X

$r := \mathrm{rank}\, S_X$

$S_X^*/S_X = (\mathbb{Z}/2\mathbb{Z})^{\oplus a}$

$$X^\iota = \begin{cases} \phi & S_X = U(2) \oplus E_8(2) \\ C^{(1)} \amalg C^{(1)} & S_X = U \oplus E_8(2) \\ C^{(g)} \amalg \mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1 & \text{otherwise} \end{cases}$$

$$g = \frac{22 - r - a}{2}, \quad \sharp\mathbb{P}^1 = \frac{r - a}{2}$$

p-elementary lattices with some conditions are classified.

- $p = 2$: V.V. Nikulin
- $p \neq 2$: A.N. Rudakov and I.R. Shafarevich

$$\text{invariants} \rightsquigarrow \begin{cases} r, a, \delta & p = 2 \\ r, a & p \neq 2 \end{cases}$$

Problem

Let σ be a non-symplectic automorphism which acts trivially on S_X.
Describe the fixed locus X^σ in terms of the invariants of p-elementary lattices.

Non-symplectic automorphisms which act trivially on S_X

Assume $\operatorname{ord}\sigma = p \geq 3$.

$$X^\sigma = C^{(g)} \amalg \mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1 \amalg \{P_1, \ldots, P_M\}$$

where P_i is an isolated point.

$$\chi(X^\sigma) = (2 - 2g) + 2 \times (\sharp\mathbb{P}^1) + M$$

We apply the topological Lefschetz formula.

$$\begin{aligned}\chi(X^\sigma) &= \sum_{i=0}^{4}(-1)^i \mathrm{tr}(\sigma^*|H^i(X, \mathbb{R})) \\ &= 1 - 0 + \mathrm{tr}(\sigma^*|S_X) + \mathrm{tr}(\sigma^*|T_X) - 0 + 1 \\ &= 2 + r - \frac{22 - r}{p - 1}\end{aligned}$$

Apply the holomorphic Lefschetz formula:

$$\sum_{i=0}^{2} \mathrm{tr}(\sigma^*|H^i(X,\mathcal{O}_X)) = \sum_j a(P_j^{u,v}) + \sum_l b(C_l)$$

$$a(P_j^{u,v}) = \frac{1}{(1-\zeta_p^u)(1-\zeta_p^v)}, \quad b(C_l) = \frac{1-g(C_l)}{1-\zeta_p} - \frac{\zeta_p C_l^2}{(1-\zeta_p)^2}.$$

Lemma

M is determined by p and r.

If p =3, 5 or 7 then

$$M = \frac{(p-2)r-2}{p-1}.$$

$\alpha := 1 + \sigma^* + \sigma^{*2} + \cdots + \sigma^{*p-1}$, $\beta := 1 - \sigma^*$

$C(X)$: chain complex of X with coefficients in $\mathbb{Z}/p\mathbb{Z}$

$\rho = \beta^i$ $(i = 1, 2, \ldots, p-1) \rightsquigarrow \rho C(X)$: chain subcomplexe.

$$\Downarrow$$

$H_q^{\rho}(X)$: Smith special homology group.

Proposition (Smith exact sequences)

- $\rho = \beta^i$, $\bar{\rho} = \beta^{p-i}$ $(i = 1, 2, \ldots, p-1)$.

$$\cdots \to H_q^{\bar{\rho}}(X) \oplus H_q(X^{\sigma}) \to H_q(X) \to H_q^{\rho}(X) \to \cdots,$$

- $\cdots \to H_q^{\alpha}(X) \to H_q^{\beta^j}(X) \to H_q^{\beta^{j+1}}(X) \to \cdots$.

By Smith exact sequences, we have the following:

Proposition

$$\sum_q \dim H_q(X^\sigma) = \frac{20 + 2p + (p-2)r - 2(p-1)a}{p-1}.$$

$$\sum_q \dim H_q(X^\sigma) - \chi(X^\sigma) = 2 \dim H_1(X^\sigma) = 4g.$$

Theorem (Nikulin, Oguiso, Zhang, Artebani, Sarti, **T**)

Assume S_X is p-elementary.

$\exists\sigma : \mathrm{ord}\, p \iff 22 - r - (p-1)a \in 2(p-1)\mathbb{Z}_{\geq 0}$

$$X^{\sigma} = C^{(g)} \amalg \mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1 \amalg \{P_1, \ldots, P_M\},$$

$$g = \frac{22 - r - (p-1)a}{2(p-1)},$$

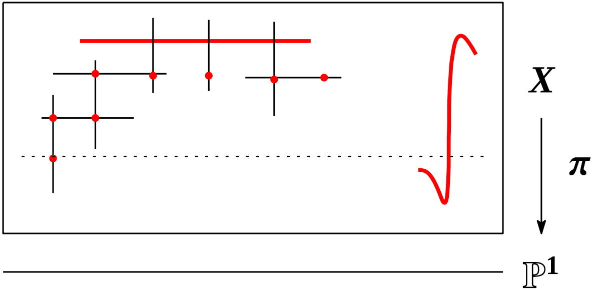
$\sharp\mathbb{P}^1$ is also determined by p, r and a.

(If p =3, 5 or 7 then $\sharp\mathbb{P}^1 = \dfrac{2 + r - (p-1)a}{2(p-1)}$.)

Example of order 7

$$X : y^2 = x^3 + x + t^7, \qquad S_X = U \oplus E_8$$

$$\sigma : (x, y, t) \longmapsto (x, y, \zeta_7 t)$$



$$X^\sigma = C^{(1)} \amalg \mathbb{P}^1 \amalg \{P_1, \ldots, P_8\}.$$

prime-power order

Theorem

σ: non-symplectic automorphism of order p^k $(k \geq 2)$.

$$X^\sigma = \mathbb{P}^1 \amalg \cdots \amalg \mathbb{P}^1 \amalg \{P_1, \ldots, P_M\}.$$

$\sharp\mathbb{P}^1$ and M are determined by p and r.

- $\operatorname{ord}\sigma = 2^k$ ··· Schütt ($\operatorname{rank} T_X = \operatorname{ord}\sigma$), T
- $\operatorname{ord}\sigma = 3^2$ ··· T
- $\operatorname{ord}\sigma = 3^3$ ··· Oguiso & Zhang
- $\operatorname{ord}\sigma = 5^2$ ··· Kondo

Problem

Study the following:

- Non-symplectic automorphisms which do NOT act trivially on S_X.
- Symplectic and non-symplectic automorphisms.
(e.g., σ : automorphism of order $\mathbf{9}$ s.t. $\sigma^* = \zeta_3\omega_X$)

Non-symplectic automorphisms of order 32

Remark

Let σ be a non-symplectic automorphism of order I.

- Φ :Euler function $\Longrightarrow \Phi(I) | \operatorname{rank} T_X \leq 21$.
- If σ acts on trivially on S_X and I is prime-power then $I = p^k = 2^\alpha, 3^\beta, 5, 5^2, 7, 11, 13, 17, 19$ $(1 \leq \alpha \leq 4, 1 \leq \beta \leq 3)$

But $\Phi(2^5) = 16$

Example (Oguiso)

$X_{\mathrm{og}} : y^2 = x^3 + t^2 x + t^{11}$

$\sigma_{\mathrm{og}}(x, y, t) = (\zeta_{32}^{18} x, \zeta_{32}^{11} y, \zeta_{32}^{2} t)$

σ_{og} is a non-symplectic automorphism of order 32 which acts on $S_{X_{\mathrm{og}}}$ as involution.

Theorem (T)

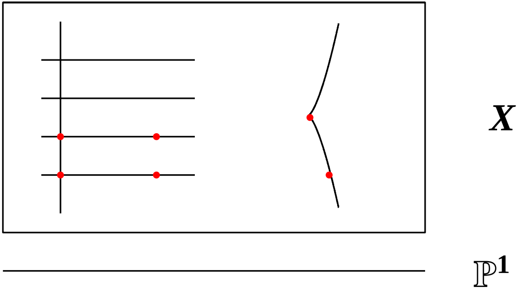
Let X be a $K3$ surface and σ a non-symplectic automorphism of order 32 on X.

(1) The fixed locus of σ has exactly six points.

(2) $(X, \langle\sigma\rangle) \simeq (X_{\mathrm{og}}, \langle\sigma_{\mathrm{og}}\rangle)$

Order 32

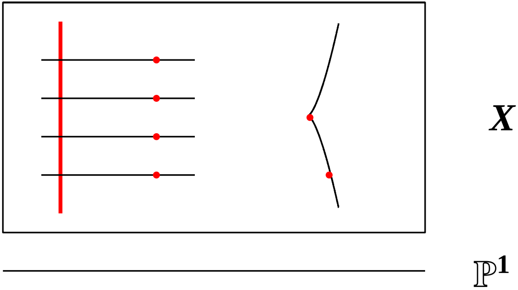
$$X : y^2 = x^3 + t^2 x + t^{11}$$
$$\sigma : (x, y, t) \mapsto (\zeta_{32}^{18} x, \zeta_{32}^{11} y, \zeta_{32}^{2} t)$$



$$X^\sigma = \{P_1, \ldots, P_6\}.$$

Order 16

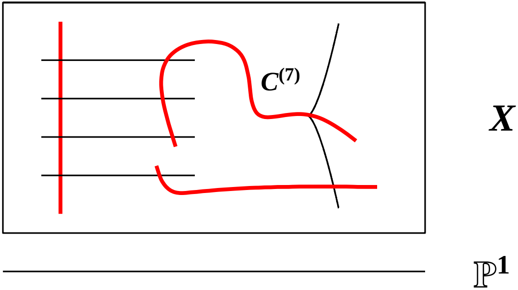
$$X : y^2 = x^3 + t^2 x + t^{11}$$
$$\sigma : (x, y, t) \mapsto (\zeta_{16}^2 x, \zeta_{16}^{11} y, \zeta_{16}^2 t)$$



$$X^\sigma = \mathbb{P}^1 \amalg \{P_1, \ldots, P_6\}.$$

Order 2

$$X : y^2 = x^3 + t^2 x + t^{11}$$
$$\sigma : (x, y, t) \mapsto (x, -y, t)$$



$$X^\sigma = C^{(7)} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1.$$